

A PRIORI ESTIMATES FOR PERIODIC SOLUTIONS TO THE MODIFIED BENJAMIN-ONO EQUATION BELOW $H^{1/2}(\mathbb{T})$

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ABSTRACT. We prove a priori estimates for real-valued periodic solutions to the modified Benjamin-Ono equation for initial data in H^s where $s > 1/4$. Our approach relies on localizing Fourier restriction spaces in time, after which one recovers the dispersive properties from Euclidean space.

1. INTRODUCTION

We will discuss periodic solutions to cubic 1d Schrödinger-like equations with derivative nonlinearity. In particular, we want to analyze the modified Benjamin-Ono equation on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$:

$$(1) \quad \begin{cases} \partial_t u(t, x) + \mathcal{H} \partial_{xx} u(t, x) &= -\partial_x(u^3)/3, \quad (t, x) \in (\mathbb{R}, \mathbb{T}), \\ u(0, x) &= u_0(x), \end{cases}$$

where we require u to be a real-valued solution. \mathcal{H} denotes the Hilbert transform, i.e.,

$$\begin{aligned} \mathcal{H} : L^2(\mathbb{T}) &\rightarrow L^2(\mathbb{T}) \\ f &\mapsto (-i \operatorname{sgn}(\xi) \hat{f}(\xi))^\vee(x) \end{aligned}$$

But in fact the following nonlinear Schrödinger-equation is also amenable to the employed methods:

$$(2) \quad \begin{cases} i \partial_t u(t, x) + \partial_{xx} u(t, x) &= i \partial_x(|u|^2 u), \quad (t, x) \in (\mathbb{R}, \mathbb{T}), \\ u(0, x) &= u_0(x). \end{cases}$$

On the real line the equations share the scaling symmetry

$$u(t, x) \rightarrow \lambda^{1/2} u(\lambda^2 t, \lambda x),$$

and consequently, the scaling critical regularity of these equations is $s_c = 0$, but it is known that uniform continuity breaks down below $s < 1/2$.

On the real line (1) has been analyzed by Guo in [6]: In [6] it was proved that the Cauchy problem given by (1) is locally wellposed with uniform continuity of the data-to-solution mapping as long as $s \geq 1/2$, see also the earlier work [13] and references therein. Furthermore, for smooth and real-valued solutions a priori estimates have been established for $s > 1/4$ in [6]. For periodic solutions global wellposedness in $H^{1/2}(\mathbb{T})$ was shown in [7].

On the real line Takaoka showed in [15] that the Cauchy problem for the derivative NLS is locally wellposed in $H^{1/2}(\mathbb{R})$ making use of the Fourier restriction spaces and a gauge transform to remedy the particularly harmful nonlinear term $|u|^2 \partial_x u$.

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Global wellposedness was later shown employing the I-method in [4]. Adapting the Fourier restriction spaces and the gauge transform to the periodic setting, Herr showed in [9] that the Cauchy problem is locally wellposed in $H^{1/2}(\mathbb{T})$, see also [5] and references therein. Takaoka showed in [16] a priori estimates for $s > 12/25$.

The purpose of this note is to show that the methods from [6] to show a priori estimates in a setting where the data-to-solution mapping fails to be uniformly continuous extend to the periodic case. The main observation is that after localization in time to small frequency dependent time intervals we can make use of the Euclidean space smoothing effects, for instance the Strichartz estimates. We are able to obtain the same regularity for a priori estimates like in Euclidean space:

Theorem 1.1. *Let u_0 be a smooth and real-valued initial datum for (1). For $1/4 < s < 1/2$, there exists $\varepsilon_s \ll 1$ so that we find the a priori estimate*

$$(3) \quad \sup_{t \in [-1, 1]} \|u(t)\|_{H^s} \leq C(\|u_0\|_{H^s})$$

to hold provided that $\|u_0\|_{H^s} \leq \varepsilon_s$.

The method we will use to show a priori estimates previously employed in [6] (see also [10] for a previous application and references therein) can be perceived as a combination of the perturbative approach and the energy method:

We employ Fourier restriction spaces to capture the dispersive effect. But in order to remedy the regularity loss through the derivative nonlinearity which becomes more and more dangerous at lower regularities and for low modulations we have to localize time on a scale antiproportional to the frequency which also requires to prove energy estimates.

Recall that for a general dispersive equation (see [17] for notation)

$$\begin{cases} i\partial_t u(t, x) + \omega(\nabla/i)u(t, x) &= F(u), \quad (t, x) \in (\mathbb{R}, \mathbb{K}), \quad \mathbb{K} \in \{\mathbb{T}, \mathbb{R}\}, \\ u(0, x) &= u_0(x), \end{cases}$$

one has the $X^{s,b}$ -energy estimate $\|\eta(t)u\|_{X^{s,b}} \lesssim \|u_0\|_{H^s} + \|F(u)\|_{X^{s,b-1}}$ for $b > 1/2$. Consequently, one has to prove a nonlinear estimate $\|F(u)\|_{X^{s,b-1}} \lesssim G(\|u\|)_{X^{s,b}}$.

Performing a localization in time on a scale antiproportional to the frequency only allows one to estimate the shorttime Fourier restriction norm $F^s(T)$ in terms of a norm $N^s(T)$ for the nonlinearity and an energy norm $E^s(T)$ which is uniform in time $t \in [-T, T]$ (see Proposition 3.2).

Therefore one also has to propagate this energy norm in terms of the shorttime Fourier restriction norm, which will be done in Proposition 6.1. As for the usual Fourier restriction norms one has to estimate the nonlinearity in the $N^s(T)$ norm in terms of the shorttime Fourier restriction norm (see Proposition 5.9).

For $s > 1/4$ we will show the bounds (cmp. [10])

$$\begin{cases} \|u\|_{F^s(T)} &\lesssim \|u\|_{E^s(T)} + \|\partial_x(u^3/3)\|_{N^s(T)} \\ \|\partial_x(u^3/3)\|_{N^s(T)} &\lesssim \|u\|_{F^s(T)}^3 \\ \|u\|_{E^s(T)}^2 &\lesssim \|u_0\|_{H^s}^2 + \|u\|_{F^{1/4+}(T)}^2 \|u\|_{F^s(T)}^4 \end{cases}$$

and the proof will be concluded by a continuity argument.

The paper is organized as follows: In Section 2 we introduce notation, in Section 3 we show how to conclude the proof with the shorttime trilinear estimate proven in Section 5. The proof of the shorttime trilinear estimate relies on shorttime linear

and bilinear estimates from Section 4, and the propagation of the energy norm will be carried out in Section 6.

2. NOTATION

Let $\eta_0 : \mathbb{R} \rightarrow [0, 1]$ denote an even smooth function, $\text{supp}(\eta_0) \subseteq [-8/5, 8/5]$, $\eta_0 \equiv 1$ on $[-5/4, 5/4]$. We will denote dyadic numbers with capital letters N, K, J and their binary logarithm with the corresponding minuscules n, k, j . For $k \in \mathbb{N}$ we set $\eta_k(\tau) = \eta_0(\tau/2^k) - \eta_0(\tau/2^{k-1})$, which gives a smooth inhomogeneous partition of unity for the modulation variable. We write $\eta_{\leq m} = \sum_{j=0}^m \eta_j$ for $m \in \mathbb{N}$. We consider unions of intervals $I_n = \{\xi \in \mathbb{R} \mid |\xi| \in [N, 2N - 1]\}$, $N = 2^n$, $n \in \mathbb{N}$ and $I_0 = [-1, 1]$. The (I_n) partition frequency space.

Also, we use the following convention for the Fourier transform

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{T}} f(x) e^{-ix\xi} dx, \quad f \in C^\infty(\mathbb{T}),$$

which is extended on $L^2(\mathbb{T})$ by common means.

We also need the Fourier transform for space-time variables:

$$\mathcal{F}f(\xi, \tau) = \tilde{f}(\xi, \tau) = \int_{\mathbb{R}} \int_{\mathbb{T}} f(x, t) e^{-it\tau} e^{-ix\xi} dx dt, \quad f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{C}.$$

We denote the Littlewood-Paley projector onto frequencies of order 2^k , $k \in \mathbb{N}_0$ with $P_k : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$, that is $(P_k u)^\wedge(\xi) = 1_{I_k}(\xi) \hat{u}(\xi)$. The dispersion relation for the Benjamin-Ono equation reads $\omega(\xi) = -\xi|\xi|$. Note that the general statements from Sections 2 and 3 do not depend on the precise form of the dispersion relation. Next, we define an $X^{s,b}$ -type space for the Fourier transform of frequency-localized functions:

$$X_k = \{f : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{C} \mid$$

$$\text{supp}(f) \subseteq I_k \times \mathbb{R}, \|f\|_{X_k} = \sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - \omega(\xi)) f(\xi, \tau)\|_{\ell_\xi^2 L_\tau^2} < \infty\}.$$

Partitioning the modulation variable through a sum over η_j yields the estimate

$$(4) \quad \left\| \int_{\mathbb{R}} |f_k(\xi, \tau')| d\tau' \right\|_{\ell_\xi^2} \lesssim \|f_k\|_{X_k}.$$

Also, we record the estimate

$$(5) \quad \begin{aligned} & \sum_{j=l+1}^{\infty} 2^{j/2} \|\eta_j(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_k(\xi, \tau')| \cdot 2^{-l} (1 + 2^{-l} |\tau - \tau'|)^{-4} d\tau'\|_{L_\tau^2 \ell_\xi^2} \\ & + 2^{l/2} \|\eta_{\leq l}(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_k(\xi, \tau')| \cdot 2^{-l} (1 + 2^{-l} |\tau - \tau'|)^{-4} d\tau'\|_{L_\tau^2 \ell_\xi^2} \\ & \lesssim \|f_k\|_{X_k} \end{aligned}$$

For a proof see [6, Equation (3.9), p. 1099].

In particular, we find for a Schwartz-function γ for $k, l \in \mathbb{N}$, $t_0 \in \mathbb{R}$, $f_k \in X_k$ the estimate

$$(6) \quad \|\mathcal{F}[\gamma(2^l(t - t_0)) \cdot \mathcal{F}^{-1}(f_k)]\|_{X_k} \lesssim_\gamma \|f_k\|_{X_k}$$

We define the following spaces:

$$E_k = \{u_0 : \mathbb{T} \rightarrow \mathbb{C} \mid P_k u_0 = u_0, \|u_0\|_{E_k} = \|u_0\|_{L^2} < \infty\},$$

which are going to be the spaces for the dyadically localized energy. Next, we define

$$C_0(\mathbb{R}, E_k) = \{u_k \in C(\mathbb{R}, E_k) \mid \text{supp}(u_k) \subseteq [-4, 4] \times \mathbb{R}\}$$

and finally, we define for a frequency 2^k the following shorttime $X^{s,b}$ -space:

$$F_k = \{u_k \in C_0(\mathbb{R}, E_k) \mid \|u_k\|_{F_k} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[u_k \eta_0(2^k(t - t_k))]\|_{X_k} < \infty\}$$

Similarly, we define the spaces to capture the nonlinearity:

$$N_k = \{u_k \in C_0(\mathbb{R}, E_k) \mid \|u_k\|_{N_k} = \sup_{t_k \in \mathbb{R}} \|(\tau - \omega(\xi) + i2^k)^{-1} \mathcal{F}[u_k \eta_0(2^k(t - t_k))]\|_{X_k} < \infty\}.$$

We localize the spaces in time in the usual way. For $T \in (0, 1]$ we set

$$F_k(T) = \{u_k \in C([-T, T], E_k) \mid \|u_k\|_{F_k(T)} = \inf_{\tilde{u}_k = u_k \text{ in } [-T, T]} \|\tilde{u}_k\|_{F_k} < \infty\}$$

and

$$N_k(T) = \{u_k \in C([-T, T], E_k) \mid \|u_k\|_{N_k(T)} = \inf_{\tilde{u}_k = u_k \text{ in } [-T, T]} \|\tilde{u}_k\|_{N_k} < \infty\}.$$

We assemble the spaces for dyadically localized frequencies in a straightforward manner by virtue of Littlewood-Paley theory: We have the following energy space for the initial data

$$E^s = \{\phi : \mathbb{T} \rightarrow \mathbb{C} \mid \|\phi\|_{E^s}^2 = \sum_{k \geq 0} 2^{2ks} \|P_k \phi\|_{L^2}^2 < \infty\},$$

and for the solution we consider

$$E^s(T) = \{u \in C([-T, T], H^\infty) \mid \|u\|_{E^s(T)}^2 = \sum_{k \geq 0} \sup_{t_k \in [-T, T]} 2^{2ks} \|P_k u(t_k)\|_{L^2}^2 < \infty\}.$$

We define the shorttime $X^{s,b}$ -space for the solution

$$F^s(T) = \{u \in C([-T, T], H^\infty) \mid \|u\|_{F^s(T)}^2 = \sum_{k \geq 0} 2^{2ks} \|P_k u\|_{F_k(T)}^2 < \infty\},$$

and for the nonlinearity we consider

$$N^s(T) = \{u \in C([-T, T], H^\infty) \mid \|u\|_{N^s(T)}^2 = \sum_{k \geq 0} 2^{2ks} \|P_k u\|_{N_k(T)}^2 < \infty\}.$$

We are also going to employ the notion of k -acceptable time multiplication factors (cf. [6, 10]): For $k \in \mathbb{N}_0$ we set

$$S_k = \{m_k \in C^\infty(\mathbb{R}, \mathbb{R}) : \|m_k\|_{S_k} = \sum_{j=0}^{10} 2^{-jk} \|\partial^j m_k\|_{L^\infty} < \infty\}.$$

The estimates (cf. [10, Eq. (2.21), p. 273])

$$(7) \quad \begin{cases} \|\sum_{k \geq 0} m_k(t) P_k(u)\|_{F^s(T)} \lesssim (\sup_{k \geq 0} \|m_k\|_{S_k}) \cdot \|u\|_{F^s(T)}, \\ \|\sum_{k \geq 0} m_k(t) P_k(u)\|_{N^s(T)} \lesssim (\sup_{k \geq 0} \|m_k\|_{S_k}) \cdot \|u\|_{N^s(T)}, \\ \|\sum_{k \geq 0} m_k(t) P_k(u)\|_{E^s(T)} \lesssim (\sup_{k \geq 0} \|m_k\|_{S_k}) \cdot \|u\|_{E^s(T)}, \end{cases}$$

are immediate.

In particular, we can assume $F_k(T)$ functions to be supported in an interval $[-T - 2^{-k-10}, T + 2^{-k-10}]$.

3. PROOF OF THEOREM 1.1

We will bootstrap the $F^s(T)$ -norm of the solution. Propositions 3.1 and 3.2 and Lemmas 3.3 and 3.4 with their proofs can be found almost verbatim in [10] in the real line case. The assertions and their proofs carry over.

In order to conclude a bound for the Sobolev norm we need the following embedding:

Proposition 3.1 ([10, Lemma 3.1., p. 274]). *Let $T \in (0, 1]$ and $u \in F^s(T)$. Then*

$$\sup_{t \in [-T, T]} \|u(t)\|_{H^s} \lesssim \|u\|_{F^s(T)}.$$

We have the following energy estimate for the $F^s(T)$ -norm:

Proposition 3.2 ([10, Proposition 3.2., p. 274]). *Let $T \in (0, 1]$ and $u, v \in C([-T, T], H^\infty)$ satisfying the equation*

$$\partial_t u + \mathcal{H} \partial_{xx} u = v \text{ in } (-T, T) \times \mathbb{T}.$$

Then we find the following estimate to hold for any $s \geq 0$:

$$\|u\|_{F^s(T)} \lesssim \|u\|_{E^s(T)} + \|v\|_{N^s(T)}$$

To carry out the bootstrap argument we need continuity and limit properties of $T' \mapsto \|u\|_{E^s(T')}$ and $T' \mapsto \|v\|_{N^s(T')}$ as $T' \rightarrow 0$.

Lemma 3.3. [10, p. 279] *Suppose that $u \in C([-T, T], H^\infty)$ and $u(0) = u_0$. Then we find the map $T' \mapsto \|u\|_{E^s(T')}$, $T' \in [0, T]$ to be increasing, continuous and*

$$\lim_{T' \rightarrow 0} \|u\|_{E^s(T')} \leq 2\|u_0\|_{H^s}.$$

We have a similar lemma for the nonlinear term.

Lemma 3.4 ([10, Lemma 4.2., p. 279]). *Assume $T \in (0, 1]$ and $v \in C([-T, T], H^\infty)$. Then the mapping $T' \mapsto \|v\|_{N^s(T')}$ is increasing and continuous on the interval $(0, T]$ and satisfies*

$$\lim_{T' \rightarrow 0} \|v\|_{N^s(T')} = 0.$$

With the above estimates at our disposal we can prove Theorem 1.1:

Proof of Theorem 1.1. Assuming that u_0 is a smooth and real-valued initial datum with $\|u_0\|_{H^s} \leq \varepsilon_s \ll 1$ ($1/4 < s < 1/2$), we find from the previous wellposedness theory the global existence of a smooth and real-valued solution $u \in C(\mathbb{R}, H^\infty)$ (see [7, Theorem 1.1]).

Because of Proposition 3.1 it will be enough to establish a bound $\|u\|_{F^s(T)} \lesssim \|u_0\|_{H^s}$ which we will prove by means of a continuity argument.

From Propositions 3.2, 5.9 and 6.1 we have for any $T' \in (0, T]$ and $s > 1/4$

$$\begin{cases} \|u\|_{F^s(T)} & \lesssim \|u\|_{E^s(T)} + \|\partial_x(u^3/3)\|_{N^s(T)} \\ \|\partial_x(u^3/3)\|_{N^s(T)} & \lesssim \|u\|_{F^s(T)}^3 \\ \|u\|_{E^s(T)}^2 & \lesssim \|u_0\|_{H^s}^2 + \|u\|_{F^{1/4+}(T)}^2 \|u\|_{F^s(T)}^4 \end{cases}$$

Next, we set $X(T') = \|u\|_{E^s(T')} + \|\partial_x(u^3)\|_{N^s(T')}$ and find the bound

$$X(T')^2 \lesssim \|u_0\|_{H^s}^2 + X(T')^6$$

by eliminating $\|u\|_{F^s(T)}$ in the above system of estimates.

Due to Lemmas 3.3, 3.4 we find that $X(T')$ is continuous and satisfies $\lim_{T' \rightarrow 0} X(T') \lesssim \|u_0\|_{H^s}$ where $\|u_0\|_{H^s} \leq \varepsilon_0$ and ε_0 is chosen sufficiently small. Finally, we deduce $X(1) \lesssim \|u_0\|_{H^s}$. \square

Remark 3.5. We only consider the defocusing case in order to be easily able to work with globally existing smooth solutions. From the proof of Theorem 1.1 it is clear that we can show a priori estimates as long as the norm of the initial data is small and we know about the existence of a (sufficiently) smooth solution. Under the assumption of the existence of a sufficiently smooth solution we can also prove a priori estimates in the focusing case.

4. SHORTTIME LINEAR AND BILINEAR ESTIMATES

In this section we are going to formulate linear and bilinear estimates for free solutions to the Schrödinger equation on \mathbb{T} . After projecting to negative and positive frequencies and applying the symmetry of motion reversal we find the estimates from below also to hold for free solutions to the Benjamin-Ono equation.

Following the heuristic that Schrödinger-waves localized in frequency around N travel with a group velocity proportional to N one expects the estimates from Euclidean space to remain true on the torus when localized to a time scale of order N^{-1} . We are going to recall shorttime Strichartz estimates (cf. [2, 8]) and a shorttime maximal function estimate (cf. [14]) and below we will prove a shorttime local smoothing estimate.

Since Strichartz estimates rely on the dispersive effect on the real line which is not available global in time on compact manifolds these estimates do not hold true on \mathbb{T} . However, localizing in time recovers the dispersive estimate and as a corollary we find the following shorttime estimates for \mathbb{T} :

Theorem 4.1. [2, Proposition 2.9, p. 583] *Suppose that $2 \leq q \leq \infty$, $2 \leq p < \infty$ and (q, p) is Schrödinger-admissible, i.e. $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$ and $u_0 \in L^2(\mathbb{T})$ with $\text{supp}(\hat{u}_0) \subseteq I_n$. Then we find the following estimate to hold:*

$$(8) \quad \|e^{it\partial_x^2} u_0\|_{L_t^q([0, 2^{-n}], L_x^p(\mathbb{T}))} \lesssim_{p,q} \|u_0\|_{L^2}$$

We also have the bilinear refinement which in the Euclidean case holds globally in time (cf. [1]):

Theorem 4.2. [8, Theorem 1.2., p. 343] *Suppose that $u_0, v_0 \in L^2(\mathbb{T})$ where $\text{supp}(\hat{u}_0) \subseteq I_n$ and $\text{supp}(\hat{v}_0) \subseteq I_k$ and $n - k \geq 4$. Then we find the following estimate to hold:*

$$(9) \quad \|e^{it\partial_x^2} u_0 e^{it\partial_x^2} v_0\|_{L_{t \in [0, 2^{-n}]}^2 L_x^2(\mathbb{T})} \lesssim 2^{-n/2} \|u_0\|_{L^2} \|v_0\|_{L^2}$$

Furthermore, we have the following maximal function estimate:

Theorem 4.3. [14, Theorem 1, p. 119] *Let $u(t, x) = e^{it\partial_x^2} u_0$ with $\text{supp}(\hat{u}_0) \subseteq I_n$. Then we find the following estimate to hold:*

$$(10) \quad \left\| \sup_{0 \leq t \leq \delta} |u| \right\|_{L_x^4} \lesssim \delta^{1/4} 2^{(n/2)+} \|u_0\|_{L^2}.$$

For short time scales we can also prove the following smoothing estimate:

Theorem 4.4. *Let $u(t, x) = e^{it\partial_x^2} u_0$ with $\text{supp}(\hat{u}_0) \subseteq I_n$. Then we find the following estimate to hold:*

$$(11) \quad \|u\|_{L_x^\infty L_t^2([0, 2^{-n}])} \lesssim 2^{-(n/2)+} \|u_0\|_{L^2}.$$

Proof. Again we can treat positive and negative frequencies separately. For the positive frequencies we write

$$u(t, x) = \sum_{k=N+1}^{2N} \hat{u}_0(k) e^{i(kx - tk^2)}, \quad \hat{u}_0(k) = a_k,$$

and consequently,

$$\begin{aligned} |u(t, x)|^2 &= \sum_{k=N+1, l=N+1}^{2N} (e^{ikx} a_k) e^{-itk^2} (e^{-ilx} a_l^*) e^{itl^2} \\ &= \sum_{k=N+1}^{2N} |a_k|^2 + \sum_{j=N+1}^{2N} \sum_{m=1}^{N-1} (e^{-ijx} a_j^*) (e^{i(j-m)x} a_{j-m}) e^{itj^2} e^{-it(j-m)^2} + h.c. \\ &= \|u_0\|_{L^2}^2 + \sum_{j=N+1}^{2N} \sum_{m=1}^{N-1} (e^{-ijx} a_j^*) (e^{i(j-m)x} a_{j-m}) e^{2itjm - itm^2} + h.c. \end{aligned}$$

Next, we are going to estimate the time integrals of the terms separately and only in terms of the absolute values of a_k (which we shall denote soon for the sake of convenience again by a_k) which allows us to deduce a bound which is uniform in x . Since the estimate of the first term is clear and the third term can be estimated like the second one we focus on the second one. After performing the time integral and taking absolute values and disposing irrelevant factors we are led to estimating the following expression:

$$\sum_{m=1}^{N-1} \sum_{j=N+1}^{2N} a_j a_{j-m} \frac{1}{m(2j-m)}$$

We change the summation variables to find the expression to be equivalent to

$$\sum_{2N+1 < J < 4N} \sum_{k+l=J, l < k} a_k a_l \frac{1}{J(k-l)} \lesssim \frac{1}{N} \sum_J \sum_{k+l=J, l < k} a_k a_l \frac{1}{(k-l)}$$

We perceive the latter expression as the following bilinear operator (again assuming the coefficients to be nonnegative):

$$\begin{aligned} T : \ell^1 \times \ell^\infty &\rightarrow \mathbb{C} \\ (a, b) &\mapsto \sum_J \sum_{\substack{k+l=J, l < k, \\ k, l \in \{N+1, \dots, 2N\}}} a_k b_l \frac{1}{k-l} \end{aligned}$$

The operator norm is computed as follows:

$$\begin{aligned} \sum_{2N+1 < J < 4N} \sum_{\substack{k=N+1, \\ 2k-J > 0}}^{2N} a_k b_{J-k} \frac{1}{2k-J} &= \sum_{k=N+1}^{2N} a_k \sum_{\substack{2N+1 < J < 4N, \\ 2k-J > 0}} b_{J-k} \frac{1}{2k-J} \\ &\lesssim \sum_{k=N+1}^{2N} a_k \|b\|_{\ell^\infty} \log(N) \lesssim \log(N) \|a\|_{\ell^1} \|b\|_{\ell^\infty} \end{aligned}$$

Likewise we find the bound $\log(N) \|a\|_\infty \|b\|_1$ from which we infer from multilinear interpolation the bound $\log(N) \|a\|_2 \|b\|_2$.

Putting everything together we arrive at

$$\|u\|_{L_x^\infty L_t^2([0, 2^{-n}])} \lesssim \log N N^{-1/2} \|u_0\|_{L^2},$$

which finishes the proof. \square

Remark 4.5. Writing a general function $u_n(t, x)$ with time support in J_n with $|J_n| \lesssim 2^{-n}$ and frequency support in I_n as superposition of free solutions we find the following estimates to hold:

$$\begin{aligned} \|u_n\|_{L_t^q L_x^p} &\lesssim \|\mathcal{F}u_n\|_{X_n} \text{ for } \frac{2}{q} + \frac{1}{p} = \frac{1}{2} \\ \|u_n\|_{L_x^4 L_{t \in J_n}^\infty} &\lesssim |I|^{1/4} 2^{(n/2)+} \|\mathcal{F}u_n\|_{X_n} \\ \|u_n\|_{L_x^\infty L_{t \in J_n}^2} &\lesssim 2^{-(n/2)+} \|\mathcal{F}u_n\|_{X_n} \end{aligned}$$

This mechanism is known as transfer principle because the properties of the free solutions are inherited. We prove the first assertion, the other ones can be proven by the same means.

We use Fourier inversion formula and compute

$$\begin{aligned} \|u_n\|_{L_t^q(\mathbb{R}, L_x^p)} &= \left\| \int_{\mathbb{R}} d\tau e^{it\tau} \sum_{k \in \mathbb{Z}} e^{ixk} \mathcal{F}u_n(\tau, k) \right\|_{L_t^q L_x^p} \\ &\sim \left\| \int_{\mathbb{R}} d\tau e^{it\tau} \sum_{k \in \mathbb{Z}} e^{ixk} \mathcal{F}u_n(\tau, k) \right\|_{L_{t \in J_n}^q L_x^p} \\ &=_{\tau = \tilde{\tau} + \omega(k)} \left\| \sum_{k \in \mathbb{Z}} e^{ixk} \int_{\mathbb{R}} d\tilde{\tau} e^{it\tilde{\tau}} e^{it\omega(k)} \mathcal{F}u_n(\tilde{\tau} + \omega(k), k) \right\|_{L_{t \in J_n}^q L_x^p} \\ &= \left\| \int_{\mathbb{R}} d\tilde{\tau} e^{it\tilde{\tau}} \sum_{k \in \mathbb{Z}} e^{ixk} e^{it\omega(k)} \mathcal{F}u_n(\tilde{\tau} + \omega(k), k) \right\|_{L_{t \in J_n}^q L_x^p} \end{aligned}$$

We set $f(t, x, \tilde{\tau}) = \sum_{k \in \mathbb{Z}} e^{ixk} e^{it\omega(k)} \mathcal{F}u_n(\tilde{\tau} + \omega(k), k)$ and denote $J_n = [a, a + c2^{-n}]$. We observe that f is a free solution for any $\tilde{\tau} \in \mathbb{R}$ and hence $\|f(t, x, \tilde{\tau})\|_{L_{t \in J_n}^q L_x^p} \lesssim \|f(a, x, \tilde{\tau})\|_{L_x^2}$. We have further from Plancherel

$$\begin{aligned} \|f(a, x, \tilde{\tau})\|_{L_x^2} &= \left\| \sum_{k \in \mathbb{Z}} e^{ixk} e^{ia\omega(k)} \mathcal{F}u_n(\tilde{\tau} + \omega(k), k) \right\|_{L_x^2} \\ &= \left(\sum_{k \in \mathbb{Z}} \left| e^{ia\omega(k)} \mathcal{F}u_n(\tilde{\tau} + \omega(k), k) \right|^2 \right)^{1/2} = \|\mathcal{F}u_n(\tilde{\tau} + \omega(k), k)\|_{\ell_k^2} \end{aligned}$$

and finally from partitioning the modulation variable, Cauchy-Schwarz and inverting the change of variables

$$\begin{aligned}
& \int_{\mathbb{R}} d\tilde{\tau} \|\mathcal{F}u_n(\tilde{\tau} + \omega(k), k)\|_{\ell_k^2} = \int_{\mathbb{R}} d\tilde{\tau} \sum_{j=0}^{\infty} \eta_j(\tilde{\tau}) \|\mathcal{F}u_n(\tilde{\tau} + \omega(k), k)\|_{\ell_k^2} \\
&= \sum_{j=0}^{\infty} \int_{\mathbb{R}} d\tilde{\tau} \eta_j(\tilde{\tau}) \|\mathcal{F}u_n(\tilde{\tau} + \omega(k), k)\|_{\ell_k^2} \\
&\lesssim \sum_{j=0}^{\infty} 2^{j/2} \left(\int_{\mathbb{R}} d\tilde{\tau} \eta_j(\tilde{\tau})^2 \sum_{k \in \mathbb{Z}} |\mathcal{F}u_n(\tilde{\tau} + \omega(k), k)|^2 \right)^{1/2} \\
&= \sum_{j=0}^{\infty} 2^{j/2} \left(\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} d\tilde{\tau} \eta_j(\tilde{\tau})^2 |\mathcal{F}u_n(\tilde{\tau} + \omega(k), k)|^2 \right)^{1/2} \\
&= \sum_{j=0}^{\infty} 2^{j/2} \left(\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} d\tau \eta_j(\tau - \omega(k))^2 |\mathcal{F}u_n(\tau, k)|^2 \right)^{1/2} = \|\mathcal{F}u_n\|_{X_n}.
\end{aligned}$$

5. SHORTTIME TRILINEAR ESTIMATES

Following the notation from [6] we set for $k \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$

$$\begin{aligned}
\dot{D}_{k,l} &= \{(\xi, \tau) \in \mathbb{Z} \times \mathbb{R} \mid \xi \in I_k, |\tau - \omega(\xi)| \sim 2^j\}, \\
\tilde{D}_{k,l} &= \{(\xi, \tau) \in \mathbb{Z} \times \mathbb{R} \mid \xi \in I_k, |\tau - \omega(\xi)| \lesssim 2^j\}.
\end{aligned}$$

With the estimates for free solutions from Section 4 at our disposal and invoking the transfer principle we work out estimates for Schrödinger wave interactions localized in frequency, modulation and time:

Proposition 5.1. *For $i \in \{1, 2, 3\}$ suppose that $f_{k_i} \in L^2(\mathbb{Z} \times \mathbb{R})$, $\text{supp}_{\xi}(f_{k_i}) \subseteq I_{k_i}$ and $\text{supp}_t(\mathcal{F}^{-1}f_{k_i}) \subseteq J_i$, where $|J_i| \lesssim 2^{-k_i}$.*

(a) *Suppose that $k_1 \leq k_2 \leq k_3 - 10, |k_3 - k_4| \leq 5$. Then we find the following estimate to hold:*

$$(12) \quad \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau, \xi}^2} \lesssim 2^{-k_4/2} 2^{k_1/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}$$

(b) *Suppose that $|k_i - k_j| \leq 5$ for any $i, j \in \{1, 2, 3, 4\}$. Then we find the following estimate to hold:*

$$(13) \quad \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau, \xi}^2} \lesssim \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}$$

(c) *Suppose that $|k_3 - k_4| \leq 5, |k_3 - k_2| \leq 5, k_1 \leq k_3 - 30, \text{supp}_t(\mathcal{F}^{-1}f_{k_1}) \subseteq J_1, |J_1| \lesssim 2^{-k_4}$. Then we find the following estimate to hold:*

$$(14) \quad \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau, \xi}^2} \lesssim 2^{(k_1+)/2} 2^{-(k_4/2)+} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}$$

Proof. For the estimate (12) we observe by Hölder ($\|1_{\tilde{D}_{k_4, j_4}}\|_{L^\infty} = 1$) and Plancherel:

$$\begin{aligned}
\|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau, \xi}^2} &\leq \|\mathcal{F}^{-1}f_{k_1} \cdot \mathcal{F}^{-1}f_{k_2} \cdot \mathcal{F}^{-1}f_{k_3}\|_{L_{t, x}^2} \\
&\leq \|\mathcal{F}^{-1}f_{k_1}\|_{L_t^\infty L_x^\infty} \|\mathcal{F}^{-1}f_{k_2} \cdot \mathcal{F}^{-1}f_{k_3}\|_{L_{t, x}^2}
\end{aligned}$$

Next, we apply Bernstein's inequality and the energy estimate (i.e. the Strichartz estimate with $q = \infty, p = 2$) on the first factor, the bilinear Strichartz estimate on

the second factor (9) and conclude with the transfer principle the bound

$$\lesssim 2^{k_1/2} \|f_{k_1}\|_{X_{k_1}} 2^{-k_4/2} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}},$$

which proves the claim.

For the estimate (13) we again employ Hölder, Plancherel and Hölder as above to find:

$$\begin{aligned} & \|1_{\tilde{D}_{k_4,j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau,\xi}^2} \leq \|\mathcal{F}^{-1} f_{k_1} \mathcal{F}^{-1} f_{k_2} \mathcal{F}^{-1} f_{k_3}\|_{L_{t,x}^2} \\ & \leq \|\mathcal{F}^{-1} f_{k_1}\|_{L_t^6 L_x^6} \|\mathcal{F}^{-1} f_{k_2}\|_{L_t^6 L_x^6} \|\mathcal{F}^{-1} f_{k_3}\|_{L_t^6 L_x^6} \\ & \lesssim \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}, \end{aligned}$$

where we concluded the proof with the shorttime Strichartz estimate (8) and an application of the transfer principle.

For the proof of (14) we again compute

$$\begin{aligned} & \|1_{\tilde{D}_{k_4,j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau,\xi}^2} \\ & \lesssim \|\mathcal{F}^{-1} f_{k_1} \mathcal{F}^{-1} f_{k_2} \mathcal{F}^{-1} f_{k_3}\|_{L_{t,x}^2} \\ & \lesssim \|\mathcal{F}^{-1} f_{k_1}\|_{L_x^4 L_t^\infty} \|\mathcal{F}^{-1} f_{k_2}\|_{L_x^4 L_t^\infty} \|\mathcal{F}^{-1} f_{k_3}\|_{L_x^\infty L_t^2} \end{aligned}$$

By virtue of the maximal function estimate and local smoothing estimate and an application of the transfer principle we conclude

$$\begin{aligned} & \lesssim 2^{-k_4/4} 2^{(k_1/2)+} \|f_{k_1}\|_{X_{k_1}} 2^{-k_4/4} 2^{(k_2/2)+} \|f_{k_2}\|_{X_{k_2}} 2^{-(k_4/2)+} \|f_{k_3}\|_{X_{k_3}} \\ & \lesssim 2^{(k_1/2)+} 2^{-(k_4/2)+} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}, \end{aligned}$$

which finishes the proof of (14). \square

Remark 5.2. Below let $\tilde{g}(\xi, \tau) = g(-\xi, -\tau)$ denote the reflected function. We note that by duality we can write

$$\begin{aligned} & \|1_{\tilde{D}_{k_4,j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau,\xi}^2} \\ & = \sup_{\|f_{k_4,j_4}\|_{L^2}=1} \|\tilde{f}_{k_4,j_4}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau,\xi}^1} \\ (15) \quad & = \sup_{\|f_{k_4,j_4}\|_{L^2}=1} \|f_{k_1}(f_{k_2} * f_{k_3} * \tilde{f}_{k_4,j_4})\|_{L_{\tau,\xi}^1} \\ & \leq \|f_{k_1}\|_{L_{\tau,\xi}^2} \sup_{\|f_{k_4,j_4}\|_{L^2}=1} \|\mathcal{F}^{-1} f_{k_2} \mathcal{F}^{-1} f_{k_3} \mathcal{F}^{-1} f_{k_4,j_4}\|_{L_{t,x}^2} \end{aligned}$$

and we stress that in the second and third line we do not have to consider absolute values by choosing a phase-corrected f_{k_4,j_4} .

The final expression is amenable to the considerations from Proposition 5.1. Let us suppose that $\mathcal{F}_t^{-1} f_{k_i} \subseteq J_i$, $|J_i| \lesssim 2^{-k_i}$, $i \in \{1, 2, 3\}$ and $k_4 \leq k_1 \leq k_2 - 30 \sim k_3$. We can also assume $\text{supp}_t \mathcal{F}^{-1} f_{k_4,j_4}$ to be of the lowest occurring order of $\text{supp}_t \mathcal{F}^{-1} f_{k_2}$

and $\text{supp}_t \mathcal{F}^{-1} f_{k_3}$. It follows a variant to (12):

$$\begin{aligned}
(15) &\leq \|\mathcal{F}^{-1} f_{k_1}\|_{L_{t,x}^2} \sup_{\|f_{k_4,j_4}\|_{L^2}=1} \|\mathcal{F}^{-1} f_{k_2} \mathcal{F}^{-1} f_{k_3}\|_{L_{t,x}^2} \|\mathcal{F}^{-1} f_{k_4,j_4}\|_{L_{t,x}^\infty} \\
&\lesssim 2^{-k_1/2} \|\mathcal{F}^{-1} f_{k_1}\|_{L_t^\infty L_x^2} 2^{-k_2/2} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \\
&\quad 2^{k_4/2} \sup_{\|f_{k_4,j_4}\|_{L^2}=1} \|\mathcal{F}^{-1} f_{k_4,j_4}\|_{L_t^\infty L_x^2} \\
&\lesssim 2^{-k_1/2} 2^{k_4/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \\
&\quad \sup_{\|f_{k_4,j_4}\|_{L^2}=1} \|f_{k_4,j_4}\|_{X_{k_4}} \\
&\lesssim 2^{-k_1/2} 2^{k_4/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} 2^{j_4/2} \sup_{\|f_{k_4,j_4}\|_{L^2}=1} \|f_{k_4,j_4}\|_{L_{\tau,\xi}^2}
\end{aligned}$$

Applying the estimates from Proposition 5.1 we derive a trilinear estimate

$$(16) \quad \|\partial_x(u^3)\|_{N^s(T)} \lesssim \|u\|_{F^s(T)}^3$$

In order to do so, we perform decompositions with respect to the frequency variable, that is we are searching for estimates

$$(17) \quad \|P_{k_4}(\partial_x(u_{k_1} v_{k_2} w_{k_3}))\|_{N_{k_4}} \lesssim \alpha(k_1, k_2, k_3, k_4) \|u_{k_1}\|_{F_{k_1}} \|v_{k_2}\|_{F_{k_2}} \|w_{k_3}\|_{F_{k_3}}.$$

Remark 5.3. We record the following possible frequency interactions: In any case we will find estimate (16) to hold for regularities $s > 1/4$. However, in the Euclidean case the only interaction which required $s > 1/4$ was the highly resonant diagonal $High \times High \times High \rightarrow High$ -interaction. Thus it seems feasible to lower the regularities in fact to $s > 0$ in the remaining cases as shown on the real line in [6].

- (i) $High \times Low \times Low \rightarrow High$ -interaction: This case will be treated in Lemma 5.4.
- (ii) $High \times High \times Low \rightarrow High$ -interaction: This case will be treated in Lemma 5.5.
- (iii) $High \times High \times High \rightarrow High$ -interaction: This case will be treated in Lemma 5.6.
- (iv) $High \times High \times Low \rightarrow Low$ -interaction: This case will be treated in Lemma 5.7.
- (v) $High \times High \times High \rightarrow Low$ -interaction: This case will be treated in Lemma 5.8.

We start with $High \times Low \times Low \rightarrow High$ -interaction:

Lemma 5.4. Suppose that $|k_3 - k_4| \leq 5, k_1 \leq k_2 \leq k_3 - 10$. Then we find estimate (17) to hold with $\alpha = 2^{k_1/2}$.

Proof. Plugging in the definitions (note that an additional localization in time does not worsen the estimate because we need only finitely many time localization functions) we find the lhs in (17) to be dominated by

$$\begin{aligned}
&\sup_{t_k \in \mathbb{R}} \|(\tau - \omega(\xi) + i2^{k_4})^{-1} 2^{k_4} \chi_{I_{k_4}}(\xi) \cdot \mathcal{F}[u_{k_1} \eta_0(2^{k_3}(t - t_k)) \eta_0(2^{k_3-2}(t - t_k))]\| \\
&\quad * \mathcal{F}[v_{k_2} \eta_0(2^{k_3-2}(t - t_k))] * \mathcal{F}[w_{k_3} \eta_0(2^{k_3-2}(t - t_k))]\|_{X_{k_4}}
\end{aligned}$$

We denote

$$\begin{aligned} f_{k_1}(\xi, \tau) &= \mathcal{F}[u_{k_1} \eta_0(2^{k_3}(t - t_k)) \eta_0(2^{k_3-2}(t - t_k))], \\ f_{k_2}(\xi, \tau) &= \mathcal{F}[v_{k_2} \eta_0(2^{k_3-2}(t - t_k))], \\ f_{k_3}(\xi, \tau) &= \mathcal{F}[w_{k_3} \eta_0(2^{k_3-2}(t - t_k))]. \end{aligned}$$

Because of the definition of the F_{k_i} -norm and (6) it will be enough to prove the following estimate:

$$(18) \quad 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \lesssim 2^{k_1/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}$$

For the low modulation we estimate

$$\begin{aligned} & 2^{k_4} \sum_{0 \leq j_4 < k_4} 2^{j_4/2} 2^{-k_4} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \\ & \leq \sum_{0 \leq j_4 < k_4} 2^{j_4/2} \|1_{\tilde{D}_{k_4, k_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \\ & \lesssim 2^{k_4/2} \|1_{\tilde{D}_{k_4, k_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2}, \end{aligned}$$

which yields the claim (because this is the first term in the sum from (18)).

Next, we prove (18) by applying estimate (12) from Proposition 5.1, which gives

$$\begin{aligned} & 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \\ & \lesssim 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} 2^{-k_4/2} 2^{k_1/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \\ & \lesssim 2^{k_1/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \end{aligned}$$

□

We estimate $High \times High \times Low \rightarrow High$ -interaction:

Lemma 5.5. *Suppose that $|k_3 - k_2| \leq 5$, $k_1 \leq k_3 - 20$, $|k_3 - k_4| \leq 5$. Then we find the estimate (17) to hold with $\alpha(k_1, k_2, k_3, k_4) = 2^{(0k_4)+2^{(k_1/2)+}}$.*

Proof. Making the same reductions as in the previous lemma we are left to prove the estimate

$$(19) \quad 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \lesssim 2^{(0k_4)+2^{(k_1/2)+}} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}$$

To do so we apply estimate (c) from Proposition 5.1 and find

$$\begin{aligned} & 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \\ & \lesssim 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} 2^{(k_1+)/2} 2^{-(k_4/2)+} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \\ & \lesssim 2^{0k_4+2^{(k_1+)/2}} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}, \end{aligned}$$

which finishes the proof. □

We estimate the in classical Fourier restriction norms most harmful contribution, that is $High \times High \times High \rightarrow High$ interaction:

Lemma 5.6. *Suppose that $|k_i - k_j| \leq 5$ for any $i, j \in \{1, 2, 3, 4\}$. Then we find the estimate (17) to hold with $\alpha(k_1, k_2, k_3, k_4) = 2^{k_4/2}$.*

Proof. As above we reduce to the estimate

$$(20) \quad 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \lesssim 2^{k_4/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}.$$

In order to prove this estimate we employ (13) from Proposition 5.1 to find:

$$\begin{aligned} & 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \\ & \lesssim 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \\ & \lesssim 2^{k_4/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}, \end{aligned}$$

which yields the claim. \square

We consider the contribution from $High \times High \times Low \rightarrow Low$ -interaction:

Lemma 5.7. *Suppose that $|k_1 - k_2| \leq 5$, $k_3 \leq k_1 - 20$, $k_4 \leq k_1 - 20$. Then we find the estimate (17) to hold with $\alpha(k_1, k_2, k_3, k_4) = (2k_1 - k_4)2^{\min(k_3, k_4)/2}$.*

Proof. Let $\gamma : \mathbb{R} \rightarrow [0, 1]$ denote a smooth function supported in $[-1, 1]$ with

$$\sum_{n \in \mathbb{Z}} \gamma^3(t - n) \equiv 1.$$

We find the lhs to be dominated by

$$\begin{aligned} & \sup_{t_k \in \mathbb{R}} \|(\tau - \omega(\xi) + i2^{k_4})^{-1} 2^{k_4} 1_{I_{k_4}}(\xi) \mathcal{F}[u_{k_1} \eta_0(2^{k_4}(t - t_k)) \eta_0(2^{k_4-2}(t - t_k))] * \\ & \mathcal{F}[v_{k_2} \eta_0(2^{k_4-2}(t - t_k))] * \mathcal{F}[w_{k_3} \eta_0(2^{k_4-2}(t - t_k))]\|_{X_{k_4}} \\ & = \sup_{t_k \in \mathbb{R}} \|(\tau - \omega(\xi) + i2^{k_4})^{-1} 2^{k_4} 1_{I_{k_4}}(\xi) \sum_{|n| \leq C2^{k_1-k_4}} \mathcal{F}[u_{k_1} \eta_0(2^{k_4-2}(t - t_k)) \gamma(2^{k_1+5}(t - t_k) - n)] * \\ & \mathcal{F}[v_{k_2} \eta_0(2^{k_4-2}(t - t_k)) \gamma(2^{k_1+5}(t - t_k) - n)] * \mathcal{F}[w_{k_3} \eta_0(2^{k_4-2}(t - t_k)) \gamma(2^{k_1+5}(t - t_k) - n)]\|_{X_{k_4}} \end{aligned}$$

We denote

$$\begin{aligned} f_{k_1}(\xi, \tau) &= \mathcal{F}[u_{k_1} \eta_0(2^{k_4}(t - t_k)) \eta_0(2^{k_4-2}(t - t_k)) \gamma(2^{k_1+5}(t - t_k) - n)], \\ f_{k_2}(\xi, \tau) &= \mathcal{F}[v_{k_2} \eta_0(2^{k_4-2}(t - t_k)) \gamma(2^{k_1+5}(t - t_k) - n)], \\ f_{k_3}(\xi, \tau) &= \mathcal{F}[w_{k_3} \eta_0(2^{k_4-2}(t - t_k)) \gamma(2^{k_1+5}(t - t_k) - n)]. \end{aligned}$$

Performing the usual reduction step for the low modulations it will be enough to prove the following estimate:

$$\begin{aligned} (21) \quad & 2^{k_1-k_4} 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \\ & \lesssim (2k_1 - k_4) 2^{\min(k_3, k_4)/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \end{aligned}$$

In order to do so we split up the sum over j_4 into the domains $k_4 \leq j_4 < 2k_1$ and $j_4 \geq 2k_1$.

For the first part we find making use of the variant of (12) from Proposition 5.1 pointed out in Remark 5.2:

$$\begin{aligned}
& 2^{k_1-k_4} 2^{k_4} \sum_{k_4 \leq j_4 \leq 2k_1} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau, \xi}^2} \\
& \lesssim 2^{k_1} \sum_{k_4 \leq j_4 \leq 2k_1} 2^{-j_4/2} 2^{j_4/2} 2^{-k_1} 2^{\min(k_3, k_4)/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \\
& \lesssim (2k_1 - k_4) 2^{\min(k_3, k_4)/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}.
\end{aligned}$$

For the second part of the sum we employ the argument from part (b) of Proposition 5.1 to find:

$$\begin{aligned}
& 2^{k_1-k_4} 2^{k_4} \sum_{j_4 \geq 2k_1} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3})\|_{L^2} \\
& \lesssim 2^{k_1} \sum_{j_4 \geq 2k_1} 2^{-j_4/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \\
& \lesssim \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}.
\end{aligned}$$

□

We turn to estimating $High \times High \times High \rightarrow Low$ -interaction.

Lemma 5.8. *Suppose that $|k_i - k_j| \leq 5$ for $i, j \in \{1, 2, 3\}$ and $k_4 \leq k_1 - 30$. Then we find the estimate (17) to hold with $\alpha(k_1, k_2, k_3, k_4) = (2k_1 - k_4)2^{0k_1+2k_4/2+}$.*

Proof. Using the same notation and reductions like in Lemma 5.7 it will be enough to prove

$$\begin{aligned}
(22) \quad & 2^{k_1-k_4} 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L^2} \\
& \lesssim (2k_1 - k_4) 2^{0k_1+2k_4/2+} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}.
\end{aligned}$$

Again we split the sum over j_4 into the two parts $k_4 \leq j_4 \leq 2k_1$ and $j_4 > k_1$.

For the first part we employ the duality argument from Remark 5.2 combined with the argument from part (c) of Proposition 5.1 to find

$$\begin{aligned}
& 2^{k_1-k_4} 2^{k_4} \sum_{k_4 \leq j_4 < 2k_1} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau, \xi}^2} \\
& \lesssim 2^{k_1} \sum_{k_4 \leq j_4 < 2k_1} 2^{-j_4/2} \|\mathcal{F}^{-1} f_{k_1}\|_{L_x^2 L_t^2} \|\mathcal{F}^{-1} f_{k_2}\|_{L_x^\infty L_t^2} \\
& \quad \|\mathcal{F}^{-1} f_{k_3}\|_{L_x^\infty L_t^4} \sup_{\|f_{k_4, j_4}\|_{L^2}=1} \|\gamma(2^{k_1+3}(t-t_k) - n) \mathcal{F}^{-1} f_{k_4, j_4}\|_{L_x^\infty L_t^4} \\
& \lesssim 2^{k_1} \sum_{k_4 \leq j_4 < 2k_1} 2^{-j_4/2} 2^{-k_1/2} \|f_{k_1}\|_{X_{k_1}} 2^{-(k_2/2)+} \|f_{k_2}\|_{X_{k_2}} 2^{(k_3+)/4} \|f_{k_3}\|_{X_{k_3}} 2^{j_4/2} 2^{(k_4+)/2} 2^{-k_1/4} \\
& \lesssim (2k_1 - k_4) 2^{0k_1+2k_4/2+} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}.
\end{aligned}$$

For the second part of the sum we apply (14) from Proposition 5.1 and find after performing the sum

$$\begin{aligned} & 2^{k_1} \sum_{j_4 \geq 2k_1} 2^{-j_4/2} \|1_{\tilde{D}_{k_4, j_4}}(f_{k_1} * f_{k_2} * f_{k_3})\|_{L_{\tau, \xi}^2} \\ & \lesssim 2^{k_1} \sum_{j_4 \geq 2k_1} 2^{-j_4/2} \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}} \\ & \lesssim \|f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{X_{k_2}} \|f_{k_3}\|_{X_{k_3}}, \end{aligned}$$

which yields the claim. \square

Consequently, we have proved the following proposition:

Proposition 5.9. *Suppose that $T \in (0, 1]$ and $u, v, w \in F^{1/4+}(T)$. Then we find the following estimate to hold:*

$$\|\partial_x(uvw)\|_{N^{1/4+}(T)} \lesssim \|u\|_{F^{1/4+}(T)} \|v\|_{F^{1/4+}(T)} \|w\|_{F^{1/4+}(T)}$$

Proof. We fix extensions $\tilde{u}, \tilde{v}, \tilde{w}$ which satisfy for any $k \in \mathbb{N}_0$

$$\|\tilde{u}\|_{F_k} \leq 2\|u\|_{F_k(T)}, \quad \|\tilde{v}\|_{F_k} \leq 2\|v\|_{F_k(T)}, \quad \|\tilde{w}\|_{F_k} \leq 2\|w\|_{F_k(T)},$$

which is possible because of the disjoint frequency supports. Since $P_k(\partial_x(\tilde{u}\tilde{v}\tilde{w}))$ is an extension of $P_k(\partial_x(uvw))$ it will be enough to prove

$$\begin{aligned} & \sum_{k \geq 0} 2^{2k(1/4)+} \|P_k(\partial_x(\tilde{u}\tilde{v}\tilde{w}))\|_{N_k}^2 \\ & \lesssim \left(\sum_{k \geq 0} 2^{2k(1/4)+} \|\tilde{u}\|_{F_k}^2 \right) \left(\sum_{k \geq 0} 2^{2k(1/4)+} \|\tilde{v}\|_{F_k}^2 \right) \left(\sum_{k \geq 0} 2^{2k(1/4)+} \|\tilde{w}\|_{F_k}^2 \right) \end{aligned}$$

To see this we decompose $\tilde{u} = \sum_{k \geq 0} P_k \tilde{u}$, $\tilde{v} = \sum_{k \geq 0} P_k \tilde{v}$ and $\tilde{w} = \sum_{k \geq 0} P_k \tilde{w}$. And we find

$$\|P_{k_4} \partial_x(\tilde{u}\tilde{v}\tilde{w})\|_{N_{k_4}} \leq \sum_{k_1, k_2, k_3 \geq 0} \|P_{k_4} \partial_x(\tilde{u}_{k_1} \tilde{v}_{k_2} \tilde{w}_{k_3})\|_{N_{k_4}}$$

Dividing up the sum into the interaction regions described in Remark 5.3 and applying the estimates from the above Lemmas 5.4 - 5.8 completes the proof. \square

6. ENERGY ESTIMATES

In order to close the iteration we have to propagate the energy norm and show an estimate of the energy norm in terms of the shorttime Fourier restriction norm, more precisely we are going to show the estimate

$$(23) \quad \|u\|_{E^s(T)}^2 \lesssim \|u_0\|_{H^s}^2 + \|u\|_{F^{1/4+}(T)}^4 \|u\|_{F^s(T)}^2$$

for small enough $\|u\|_{E^s(T)}$. The same estimate was proved on the real line in [6, Proposition 8.1., p. 1124].

Proposition 6.1. *Suppose that $T \in (0, 1]$ and $u \in C([-T, T], H^\infty)$ is a real-valued solution to (1). Then, for $s > 1/4$, there exists $\delta_0 > 0$ such that we find (23) to hold provided that*

$$(24) \quad \|u\|_{E^s(T)} \leq \delta_0.$$

In order to prove Proposition 6.1 we are going to employ a variant of the I-method, that is considering energy estimates after integration by parts in the time variable. In the context of shorttime norms this strategy was previously employed in the works by Koch and Tataru in [11, 12] and in [6]. The following considerations resemble the arguments on the real line from [6]. We have included the main arguments for the sake of completeness. In fact, we see from the proof that one can treat the Euclidean and periodic case simultaneously.

We will also make use of the following definition from [11]:

Definition 6.2. Let $\varepsilon > 0$ and $s \in \mathbb{R}$. Then S_ε^s is the set of real-valued spherically symmetric and smooth functions (symbols) with the following properties:

(i) symbol regularity,

$$|\partial^\alpha a(\xi)| \lesssim a(\xi)(1 + \xi^2)^{-\alpha/2},$$

(ii) decay at infinity, for $|\xi| \gg 1$ we have

$$s \leq \frac{\log a(\xi)}{\log(1 + \xi^2)} \leq s + \varepsilon, \quad s - \varepsilon \leq \frac{d \log a(\xi)}{d \log(1 + \xi^2)} \leq s + \varepsilon.$$

We note that since a and expressions involving a are going to act as a Fourier multiplier for periodic functions the actual relevant domain of a is \mathbb{Z} . However, in order to derive favourable pointwise estimates we use extended versions to the real line. Also note that if we only wanted to control the H^s -norm of u we would just had to take into account the symbols $a(\xi) = (1 + \xi^2)^s$. But since we are considering estimates uniform in time we have to allow a slightly larger class of symbols. Next, suppose that u is a smooth and real-valued solution to (1). We are going to analyze the following generalized energy:

$$E_0(u) = \int_{\xi_1 + \xi_2 = 0} a(\xi_1) \hat{u}(\xi_1) \hat{u}(\xi_2)$$

The following computations can be found almost verbatim in [6] again with the difference that the computations in [6] were carried out for a continuous frequency range.

We use the following notation for the $d - 1$ -dimensional hyperplane in d -dimensional space:

$$\Gamma_d = \{\xi_1 + \xi_2 + \dots + \xi_d = 0\}$$

and find for the derivative of $E_0(u)$ after symmetrization

$$\begin{aligned} \frac{d}{dt} E_0(u) &= R_4(u) \\ &= \frac{1}{2} \int_{\Gamma_4} i[\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)] \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) \end{aligned}$$

Next, we consider the correction term

$$E_1(u) = \int_{\Gamma_4} b_4(\xi_1, \xi_2, \xi_3, \xi_4) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4),$$

where we require the multiplier b_4 to satisfy the following identity on Γ_4 :

$$(\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4)) b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{-i}{2} (\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4))$$

And consequently, we achieve a cancellation

$$\begin{aligned} \frac{d}{dt}(E_0(u) + E_1(u)) &= R_6(u) \\ &= C \int_{\Gamma_6} b_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6) \prod_{i=1}^6 \hat{u}(\xi_i) \end{aligned}$$

We have the following proposition on choosing the multiplier b_4 smooth and smoothly extending it off diagonal, which will allow us to separate variables easier. We follow the arguments in [12] and [3].

Proposition 6.3. *Assume that $a \in S_\varepsilon^s$. Then for each dyadic $\lambda \leq \beta \leq \mu$ there is an extension of b_4 from the diagonal set*

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4 : |\xi_1| \sim \lambda, |\xi_2| \sim \beta, |\xi_3|, |\xi_4| \sim \mu\}$$

to the full dyadic set

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : |\xi_1| \sim \lambda, |\xi_2| \sim \beta, |\xi_3|, |\xi_4| \sim \mu\}$$

which satisfies

$$|b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim a(\mu)\mu^{-1}$$

and

$$|\partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \partial_4^{\alpha_4} b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim_\alpha a(\mu)\mu^{-1} \lambda^{-\alpha_1} \beta^{-\alpha_2} \mu^{-(\alpha_3 + \alpha_4)}.$$

with the implicit constant depending on α , but not on λ, β, μ .

Proof. In the following we can assume that $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) \neq 0$ as long as we show b_4 to be smooth because it is easy to see that $\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4) = 0$, whenever $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = 0$.

Furthermore, due to symmetry we can assume that $\xi_3 > 0, \xi_4 < 0$. First, we check the cases $|\xi_2| \ll |\xi_3|, |\xi_1| \ll |\xi_3|$.

Suppose that $\xi_1, \xi_2 > 0$. In this case we have $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = -2(\xi_1 \xi_2 + (\xi_1 + \xi_2)\xi_3)$ and we consider

$$Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1 \xi_2 + (\xi_2 + \xi_1)\xi_3} + \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_1 \xi_3)}$$

The size and regularity properties of the first term follow from the size and regularity properties of a . For the second term we multiply with $1 = -(\xi_1 + \xi_2)/(\xi_3 + \xi_4)$. We set

$$q(\xi, \eta) = \frac{\xi a(\xi) + \eta a(\eta)}{\xi + \eta},$$

which is a smooth function. Since q satisfies the bounds $|q| \lesssim a(N)$ and $|\partial_\xi^a \partial_\eta^b q| \lesssim a(N)N^{-(a+b)}$ for $|\xi| \sim |\eta| \sim N$, the conclusion follows also for the second term

$$\frac{(\xi_1 + \xi_2)(\xi_3 a(\xi_3) + \xi_4 a(\xi_4))}{(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_1 \xi_3)(\xi_3 + \xi_4)} = \frac{\xi_1 + \xi_2}{\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_1 \xi_3} q(\xi_3, \xi_4)$$

In the case $\xi_1 < 0, \xi_2 > 0$ we find $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = -2(\xi_1 + \xi_2)(\xi_1 + \xi_3)$. Hence,

$$\begin{aligned} Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{(\xi_1 + \xi_3)(\xi_1 + \xi_2)} + \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{(\xi_1 + \xi_3)(\xi_3 + \xi_4)} \\ &= \frac{1}{\xi_1 + \xi_3} q(\xi_1, \xi_2) - \frac{1}{\xi_1 + \xi_3} q(\xi_3, \xi_4), \end{aligned}$$

which satisfies the required bounds because $|\xi_1| \ll |\xi_3|$.

In case $|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim |\xi_4|$ we can assume $\xi_4 < 0, \xi_2 < 0$ and $\xi_1, \xi_3 > 0$ and write

$$\begin{aligned} Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{a(\xi_1)\xi_1 + a(\xi_2)\xi_2}{(\xi_1 + \xi_2)(\xi_2 + \xi_3)} + \frac{a(\xi_3)\xi_3 + a(\xi_4)\xi_4}{(\xi_1 + \xi_2)(\xi_2 + \xi_3)} \\ &= \frac{q(\xi_1, \xi_2) - q(\xi_3, -\xi_1 - \xi_2 - \xi_3)}{\xi_2 + \xi_3} \\ &= \frac{q(\xi_1, \xi_2) - q(\xi_1 + (\xi_2 + \xi_3), \xi_2 - (\xi_2 + \xi_3))}{\xi_2 + \xi_3}. \end{aligned}$$

Now the bounds follow from the size and regularity of q . \square

After smoothly extending the symbol at a dyadic scale $\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4 : |\xi_1| \sim \lambda, |\xi_2| \sim \beta, |\xi_3|, |\xi_4| \sim \mu\}$ off diagonal we can separate variables without restriction:

$$(25) \quad b_4(\xi_1, \xi_2, \xi_3, \xi_4) = b_4(N_1, N_2, N_3, N_4) \chi_1(\xi_1) \chi_2(\xi_2) \chi_3(\xi_3) \chi_4(\xi_4)$$

with bump functions χ of size $\lesssim 1$ localized at $|\xi_i| \lesssim N_i$, so that we can absorb the bump functions into the frequency projectors and return to position space. For details on the separation of variables see e.g. [12, Lemma 6.4.]. We can estimate the boundary term $E_1(u)$ in a favourable way:

Proposition 6.4. *Suppose that $a \in S_\varepsilon^s$ and $s \geq \varepsilon$. Then we have*

$$|E_1(u(t))| \lesssim \|u(t)\|_{H^{0+}}^2 E_0(u(t)).$$

Proof. We use a dyadic decomposition of Γ_4 and the expansion (25) to write

$$\begin{aligned} |E_1(u)| &= \left| \int_{\Gamma_4} b_4(\xi_1, \xi_2, \xi_3, \xi_4) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) d\Gamma_4 \right| \\ &\leq \sum_{N_1 \leq N_2 \leq N_3 \sim N_4} \left| \int_{\Gamma_4: |\xi_i| \sim N_i} b_4(\xi_1, \xi_2, \xi_3, \xi_4) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) d\Gamma_4 \right| \\ &\lesssim \sum_{N_1 \leq N_2 \leq N_3 \sim N_4} |b_4(N_1, N_2, N_3, N_4)| \left| \int_{\mathbb{T}} P_{n_1} u P_{n_2} u P_{n_3} u P_{n_4} u dx \right|. \end{aligned}$$

The size estimate of b_4 and applications of Hölder's and Bernstein's inequality imply

$$\begin{aligned} (6) &\lesssim \sum_{N_1 \leq N_2 \leq N_3 \sim N_4} a(N_4) N_4^{-1} \|P_{n_1} u\|_{L_x^\infty} \|P_{n_2} u\|_{L_x^\infty} \|P_{n_3} u\|_{L_x^2} \|P_{n_4} u\|_{L_x^2} \\ &\lesssim \sum_{N_1 \leq N_2 \leq N_3 \sim N_4} a(N_4) N_4^{-1} N_1^{1/2} N_2^{1/2} \|P_{n_1} u\|_{L_x^2} \|P_{n_2} u\|_{L_x^2} \|P_{n_3} u\|_{L_x^2} \|P_{n_4} u\|_{L_x^2} \\ &\lesssim \|u\|_{H^{0+}}^2 E_0(u), \end{aligned}$$

which yields the claim. \square

Now we estimate the remainder. The proof follows essentially the real line pendant [6, Proposition 8.5., p. 1127]:

Proposition 6.5. *Suppose that $a \in S_\varepsilon^s$, $s \geq \varepsilon$ and $T \in (0, 1]$. Then we find the following estimates to hold:*

$$\left| \int_0^T R_6(u) \right| \lesssim \|u\|_{F^{1/4+2\varepsilon}(T)}^4 \|u\|_{F^s(T)}^2, \quad \left| \int_{-T}^0 R_6(u) \right| \lesssim \|u\|_{F^{1/4+2\varepsilon}(T)}^4 \|u\|_{F^s(T)}^2$$

Proof. First we fix an extension $\tilde{u} \in C_0(\mathbb{R}, H^\infty)$ satisfying the bounds $\|P_k \tilde{u}\|_{F_k} \leq 2\|P_k u\|_{F_k(T)}$.

It will be enough to prove

$$\left| \int_0^T R_6(\tilde{u}) \right| \lesssim \|\tilde{u}\|_{F^{1/4+2\varepsilon}}^4 \|\tilde{u}\|_{F^s}^2$$

To simplify notation we write again $\tilde{u} = u$.

Symmetrization allows us to rewrite

$$R_6(u) \sim \int_{\Gamma_6} [b_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6) - b_4(-\xi_4, -\xi_5, -\xi_6, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6)] \prod_{j=1}^6 \hat{u}(\xi_j).$$

Partitioning the frequencies into dyadic blocks we use the notation $|\xi_j| \sim 2^{k_j} = K_j$ and because of symmetry we can assume that $K_1 \leq K_2 \leq K_3$, $K_4 \leq K_5 \leq K_6$. We will also write $\xi_{456} = \xi_4 + \xi_5 + \xi_6$. Because of otherwise impossible frequency interaction the largest frequency denoted with K_{\max} and the second to largest K_{sub} must be comparable.

Slicing up momentum space into dyadic blocks we find

$$(26) \quad \left| \int_0^T R_6(u) dt \right| \lesssim \sum_{K_j} \left| \int_0^T \int_{\Gamma_6: |\xi_i| \sim K_i} [b_4(\xi_1, \xi_2, \xi_3, \xi_{456})\xi_{456} - b_4(-\xi_4, -\xi_5, -\xi_6, \xi_{456})\xi_{456}] \prod_{j=1}^6 \hat{u}(\xi_j) dt \right|$$

In order to derive estimates in terms of the shorttime norms we have to localize time with bump functions supported on intervals of length antiproportional to the highest occuring frequency. Therefore let $\gamma : \mathbb{R} \rightarrow [0, 1]$ denote a nonnegative smooth function supported in $[-1, 1]$ with

$$\sum_{n \in \mathbb{Z}} \gamma^6(x - n) \equiv 1, \quad x \in \mathbb{R}.$$

Next, we bound the dyadically localized expression (26) in several cases:

In the case $K_5 \sim K_6 \sim K_{\max}$, $K_3 \lesssim K_5$ we write $C_1 = \{(K_1, \dots, K_6) : K_5 \sim K_6 \sim K_{\max}, K_3 \lesssim K_5\}$ and find for this part of (26)

$$(27) \quad \sum_{K_j \in C_1} \sum_{|n| \leq C2^{k_6}} \left| \int_{\mathbb{R}} \int_{\Gamma_6} [b_4(\xi_1, \xi_2, \xi_3, \xi_{456}) - b_4(-\xi_4, -\xi_5, -\xi_6, \xi_{456})]\xi_{456} \right. \\ \left. [\gamma(2^{k_6}t - n)1_{[0, T]} \hat{u}_{k_1}(\xi_1)] \prod_{j=2}^6 [\gamma(2^{k_6}t - n)\hat{u}_{k_j}(\xi_j)] dt \right|$$

The cases when the bump function is cut off by $1_{[0, T]}(t)$ need one additional step. We denote this exceptional set by

$$A = \{n : \forall t \in \mathbb{R} : \gamma(2^{k_6}t - n)1_{[0, T]}(t) \notin \{\gamma(2^{k_6}t - n), 0\}\}$$

and observe that $|A| \leq 4$.

Due to Proposition 6.3 we can expand b as in (25). We can absorb the Schwartz

functions into the frequency projectors and switch back to position space which reduces the problem to estimate

$$\sum_{K_j \in C_1} \sum_{|n| \lesssim 2^{k_6}, n \in A^c} (a(K_3)K_3^{-1} + a(K_6)K_6^{-1})K_3 \left| \int_{\mathbb{R}} dt \int_{\mathbb{T}} dx \prod_{j=1}^6 P_{k_j} [u\gamma(2^{k_6}t - n)] \right|$$

Applying the local smoothing estimate on $P_{k_5}u$ and $P_{k_6}u$ and the maximal function estimate on the remaining factors via the transfer principle we find from Hölder's inequality

$$\begin{aligned} (28) \quad & \left| \int_{\mathbb{R}} dt \int_{\mathbb{T}} dx \prod_{j=1}^6 P_{k_j} [u\gamma(2^{k_6}t - n)] \right| \\ & \lesssim \prod_{j=1}^4 \|P_{k_j} [u\gamma(2^{k_6}t - n)]\|_{L_x^4 L_t^\infty} \|P_{k_5} [u\gamma(2^{k_6}t - n)]\|_{L_x^\infty L_t^2} \|P_{k_6} [u\gamma(2^{k_6}t - n)]\|_{L_x^\infty L_t^2} \\ & \lesssim K_5^{-(1+\varepsilon)} \prod_{j=1}^4 K_j^{1/4+\varepsilon} \prod_{j=1}^6 \|\mathcal{F}[P_{k_j} u\gamma(2^{k_6}t - n)]\|_{X_{k_j}} \\ & \lesssim K_5^{-(1+\varepsilon)} \prod_{j=1}^4 K_j^{1/4+\varepsilon} \prod_{j=1}^6 \|P_{k_j} u\|_{F_{k_j}} \end{aligned}$$

Applying this bound on (27) $|A^c| \leq C2^{k_6}$ times we conclude

$$\begin{aligned} & \sum_{K_j: (K_j) \in C_1} (a(K_3)K_3^{-1} + a(K_6)K_6^{-1})K_5^{-\varepsilon} K_3 \prod_{j=1}^4 K_j^{1/4+\varepsilon} \prod_{j=1}^6 \|P_{k_j} u\|_{F_{k_j}} \\ & \lesssim \|u\|_{F^{1/4+2\varepsilon}}^4 \|u\|_{F^s}^2 \end{aligned}$$

For $n \in A$ we absorb the cutoff into the first factor as in (27) and noting

$$\|P_{n_1} [u1_{[0,T]}(t)\gamma(2^{k_6}t - n)]\|_{L_x^4 L_t^\infty} \leq \|P_{k_1} [u\gamma(2^{k_6}t - n)]\|_{L_x^4 L_t^\infty}$$

we can proceed as in (28)

$$\begin{aligned} & \sum_{K_j \in C_1} \sum_{n \in A} (a(K_3)K_3^{-1} + a(K_6)K_6^{-1})K_3 K_5^{-(1+\varepsilon)} \prod_{j=1}^4 K_j^{1/4+\varepsilon} \\ & \prod_{j=1}^6 \|\mathcal{F}[P_{k_j} u\gamma(2^{k_6}t - n)]\|_{X_{k_j}} \lesssim \|u\|_{F^\varepsilon}^4 \|u\|_{F^s}^2, \end{aligned}$$

which finishes the estimates in this case.

The case $K_2 \sim K_3 \sim K_{\max}$, $K_6 \lesssim K_2$ is identical to the above one after noting the symmetry $(K_1, K_2, K_3) \leftrightarrow (K_4, K_5, K_6)$.

In the case $K_3 \sim K_6$, $K_2, K_5 \ll K_6$ we find for the symbol

$$|b_4(\xi_1, \xi_2, \xi_3, \xi_{456}) - b_4(-\xi_4, -\xi_5, -\xi_6, \xi_{456})| |\xi_{456}| \lesssim a(K_3) + a(K_6).$$

Arguing like above where the roles of K_{\max} and K_{sub} are played by K_3 and K_6 this time finishes the proof. \square

To conclude the proof of the energy estimate we will derive a bound for thresholds of the frequency localized energy. We have the following lemma on frequency localized energy thresholds:

Lemma 6.6. [11, Lemma 5.5., p. 34] *For any $u_0 \in H^s(\mathbb{T})$ and $\varepsilon > 0$ there is a sequence $(\beta_n)_{n \in \mathbb{N}_0}$ satisfying the following conditions:*

- (a) $2^{2ns} \|P_n u_0\|_{L^2}^2 \leq \beta_n \|u_0\|_{H^s}^2$,
- (b) $\sum_n \beta_n \lesssim 1$,
- (c) (β_n) satisfies a log-Lipschitz condition, that is

$$|\log_2 \beta_n - \log_2 \beta_m| \leq \frac{\varepsilon}{2} |n - m|.$$

We remark that due to the construction in [11] one can always assume $\beta_0 \leq 1$. Now we can conclude the proof of Proposition 6.1.

Proof of Proposition 6.1. Let $k_0 \in \mathbb{N}_0$ and let (β_n) be an envelope sequence from Lemma 6.6 for the respective initial data u_0 and $\varepsilon \leq 1/k_0$. We are going to prove

$$(29) \quad \sup_{t \in [-T, T]} 2^{2ks} \|P_k u(t)\|_{L^2}^2 \lesssim \beta_k (\|u_0\|_{H^s}^2 + \|u\|_{F^{1/4+}(T)}^4 \|u\|_{F^s(T)}^2)$$

from which we deduce (23) after summing over k in view of property (b) from Lemma 6.6. Following the remark after Lemma 6.6 we assume $\beta_0 \leq 1$ without any restriction.

Setting $a_k = 2^{2ks} \max(1, \beta_{k_0}^{-1} 2^{-\varepsilon|k-k_0|})$ we compute the bound

$$\begin{aligned} \sum_{k \geq 0} a_k \|P_k u_0\|_{L^2}^2 &\leq \sum_k 2^{2ks} \|P_k u_0\|_{L^2}^2 + 2^{2ks} \beta_{k_0}^{-1} \|P_k u_0\|_{L^2}^2 \\ &\leq \|u_0\|_{H^s}^2 + \beta_{k_0}^{-1} \|u_0\|_{H^s}^2 \leq 3 \|u_0\|_{H^s}^2, \end{aligned}$$

when the estimate of the second term follows from the log-Lipschitz condition on β_k . In fact, we find $\max\{|\beta_k|, |\beta_k|^{-1}\} \leq 2^{k\varepsilon/2}$ from $\beta_0 \leq 1$.

We can find a function $a(\xi) \in S_\varepsilon^s$ so that

$$a(\xi) \sim a_k, \quad |\xi| \sim 2^k.$$

Next, following the computations from the beginning of this section we consider the associated energy E_0 , the boundary term E_1 and the multilinear correction term R_6 and applications of Proposition 6.3 and 6.4 give rise to the estimates

$$(30) \quad \begin{aligned} |E_0(u(T))| &\leq |E_0(u_0)| + |E_1(u_0)| + |E_1(u(T'))| + \left| \int_0^{T'} R_6(u) \right| \\ &\lesssim (1 + \delta_0^2) \|u_0\|_{H^s}^2 + \delta_0^2 |E_0(u(T'))| + \|u\|_{F^{1/4+}(T')}^4 \|u\|_{F^s(T')}^2 \end{aligned}$$

where we have used the smallness condition $\|u\|_{E^s(T)} \lesssim \delta_0 \ll 1$. Consequently,

$$(31) \quad \sup_{t \in [0, T]} |E_0(u(t))| \lesssim \|u_0\|_{H^s}^2 + \|u\|_{F^{1/4+2\varepsilon}(T)}^4 \|u\|_{F^s(T)}^2$$

and similarly for the interval $[-T, 0]$.

We find from (31) and the definition of E_0

$$\sum_{k \geq 0} a_k \|P_k u(t)\|_{L^2}^2 \lesssim \|u_0\|_{H^s}^2 + \|u\|_{F^{1/4+2\varepsilon}(T)}^4 \|u\|_{F^s(T)}^2$$

and taking $k = k_0$ proves (29) and hence finishes the proof. \square

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